A Review of Spatial (and Spatio-Temporal) Autoregressive Models

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Collaborators

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Goals

Talk based on:

- Ver Hoef, J.M., Peterson, E. E., Hooten, M. B., Hanks, E. M., and Fortin, M.-J. 2018. Spatial Autoregressive Models for Statistical Inference from Ecological Data. *Ecological Monographs*, **88**: 36–59. DOI:10.1002/ecm.1283
- Ver Hoef, J.M., Hanks, E.M., and Hooten, M.B. On the relationship between conditional (CAR) and simultaneous (SAR) autoregressive models. arXiv:1710.07000[math.ST] and in revision to *Spatial Statistics*.
- Spatio-temporal is new, but partly based on Brown, D.A. and McMahan, C.S. Sampling Strategies for Fast Updating of Gaussian Markov Random Fields. arXiv:1702.05518[stat.CO]

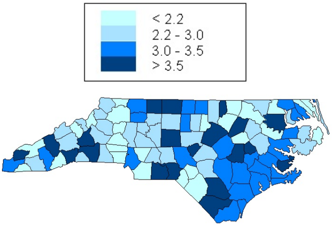
Typical Inferences

- Model Comparison and Selection (Averaging)
- Regression (Effect of Explanatory Varibles)
- Estimation of Autocorrelation Parameters
- Estimation of Other Network Parameters
- Prediction of Unsampled Locations
- Smoothing over Sampled and Unsampled Locations

Autoregressive Models vs. Geostatistics

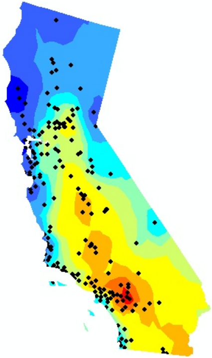
Autoregressive Models

Sudden Infant Death Syndrome



Geostatistics

Ozone



CAR and SAR defined

SAR

$$\mathbf{z} = \mathbf{B}\mathbf{z} + \boldsymbol{\nu}$$
$$\mathbf{z} - \mathbf{B}\mathbf{z} = \boldsymbol{\nu}$$
$$\mathbf{z} = (\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\nu}$$

CAR

$$Z_i|\mathbf{z}_{-i} \sim \mathrm{N}\left(\sum_{j=1}^{n} c_{ij}z_j, m_{ii}\right)$$

CAR and SAR in Linear Model

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{z} + \varepsilon; \quad E(\varepsilon) = 0, \mathrm{var}(\varepsilon) = \sigma^2_\varepsilon \mathbf{I}$$

SAR

$$\mathrm{var}(\mathbf{z}) = \mathbf{\Sigma} \equiv \sigma^2_Z((\mathbf{I} - \mathbf{B})\mathbf{\Lambda}(\mathbf{I} - \mathbf{B}'))^{-1}$$
$$\mathbf{\Sigma} = \sigma^2_Z((\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}'))^{-1}$$
$$\mathbf{\Sigma}^{-1} = ((\mathbf{I} - \rho\mathbf{W})(\mathbf{I} - \rho\mathbf{W}'))/\sigma^2_Z$$

CAR

$$\mathrm{var}(\mathbf{z}) = \mathbf{\Sigma} \equiv \sigma^2_Z(\mathbf{I} - \mathbf{C})^{-1}\mathbf{M}$$
$$\mathbf{\Sigma} = \sigma^2_Z(\mathbf{M}^{-1} - \rho\mathbf{W})^{-1}$$
$$\mathbf{\Sigma}^{-1} = (\mathbf{M}^{-1} - \rho\mathbf{W})/\sigma^2_Z$$

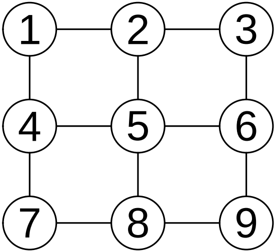
Geostatistics

$$\mathrm{var}(\mathbf{z}) = \mathbf{\Sigma}_{i,j} = \sigma^2_Z \exp(-d_{i,j}/\rho)$$
$$\mathbf{\Sigma} = \sigma^2_Z \exp(-\mathbf{D}/\rho)$$

Partial Correlation

$$\rho(Z_i, Z_j) = \frac{-\mathbf{\Sigma}^{-1}[i,j]}{\sqrt{\mathbf{\Sigma}^{-1}[i,i]\mathbf{\Sigma}^{-1}[j,j]}}$$

Simple Example



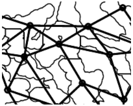
$$
\mathbf{W} = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}
$$

Covariance Matrix of Simple Example

$$\boldsymbol{\Sigma} = (\mathbf{I} - 0.2\mathbf{W})^{-1} =$$

$$\begin{pmatrix} 1.10 & 0.26 & 0.06 & 0.26 & 0.12 & 0.04 & 0.06 & 0.04 & 0.02 \\ 0.26 & 1.16 & 0.26 & 0.12 & 0.29 & 0.12 & 0.04 & 0.07 & 0.04 \\ 0.06 & 0.26 & 1.10 & 0.04 & 0.12 & 0.26 & 0.02 & 0.04 & 0.06 \\ 0.26 & 0.12 & 0.04 & 1.16 & 0.29 & 0.07 & 0.26 & 0.12 & 0.04 \\ 0.12 & 0.29 & 0.12 & 0.29 & 1.24 & 0.29 & 0.12 & 0.29 & 0.12 \\ 0.04 & 0.12 & 0.26 & 0.07 & 0.29 & 1.16 & 0.04 & 0.12 & 0.26 \\ 0.06 & 0.04 & 0.02 & 0.26 & 0.12 & 0.04 & 1.10 & 0.26 & 0.06 \\ 0.04 & 0.07 & 0.04 & 0.12 & 0.29 & 0.12 & 0.26 & 1.16 & 0.26 \\ 0.02 & 0.04 & 0.06 & 0.04 & 0.12 & 0.26 & 0.06 & 0.26 & 1.10 \end{pmatrix},$$

Neighborhood Matrix

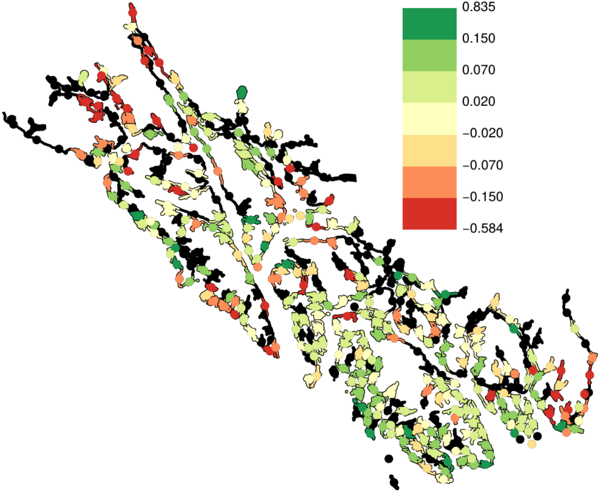


often $\rightarrow$ $\mathbf{W} = \begin{pmatrix} 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 1 \\ \vdots & & & & & \end{pmatrix}$

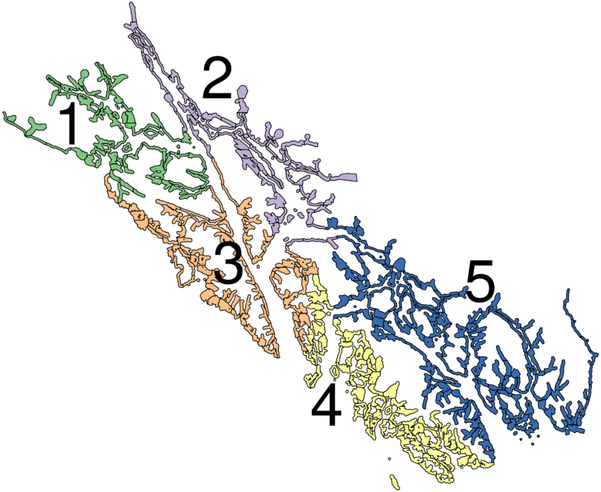
Row Standardization: $\mathbf{W}_{+} = \begin{pmatrix} 0 & 0 & 1/\mathbf{W}_{1,+} & 0 & \ldots & 0 \\ 0 & 0 & 1/\mathbf{W}_{2,+} & 0 & \ldots & 1/\mathbf{W}_{2,+} \\ \vdots & & & & & \end{pmatrix}$

- For Row Standardization, need to fix up **M** for CAR models
- $\mathbf{W}$: $1/\lambda_{[1]} < \rho < 1/\lambda_{[N]}$, $\mathbf{W}_{+}$: $-1 < \rho < 1$
- More complex structures for **W** (distance, connectivity,...)
- IAR special case of CAR using $\mathbf{W}_{+}$ and $\rho = 1$
- Estimation: MLE, Bayes MCMC, etc.

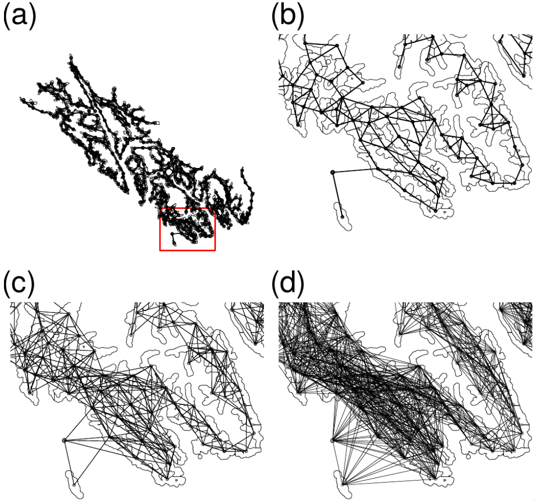
Study Area - Harbor Seal Trends



Harbor Seal Stocks



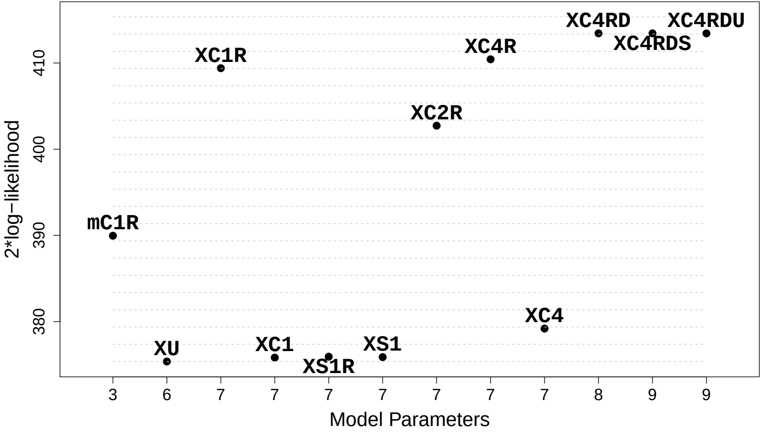
Neighbors



Models

Model Code	Fixed Effects	Covariance Model	No. Parms
mU	$\mathbf{1}$	$\sigma_\varepsilon^2 \mathbf{I}$	2
mC1R	$\mathbf{1}$	$\sigma_Z^2(\mathbf{I} - \rho[\mathbf{W}_1]_+)^{-1}[\mathbf{M}]_+$	3
XU	$\mathbf{X}_{\text{stock}}$	$\sigma_\varepsilon^2 \mathbf{I}$	6
XC1R	$\mathbf{X}_{\text{stock}}$	$\sigma_Z^2(\mathbf{I} - \rho[\mathbf{W}_1]_+)^{-1}[\mathbf{M}]_+$	7
XC1	$\mathbf{X}_{\text{stock}}$	$\sigma_Z^2(\mathbf{I} - \rho\mathbf{W}_1)^{-1}$	7
XS1R	$\mathbf{X}_{\text{stock}}$	$\sigma_Z^2[(\mathbf{I} - \rho[\mathbf{W}]_+)(\mathbf{I} - \rho[\mathbf{W}]_+)]^{-1}$	7
XS1	$\mathbf{X}_{\text{stock}}$	$\sigma_Z^2[(\mathbf{I} - \rho\mathbf{W}_1)(\mathbf{I} - \rho\mathbf{W}_1)]^{-1}$	7
XC2R	$\mathbf{X}_{\text{stock}}$	$\sigma_Z^2(\mathbf{I} - \rho[\mathbf{W}_2]_+)^{-1}[\mathbf{M}]_+$	7
XC4R	$\mathbf{X}_{\text{stock}}$	$\sigma_Z^2(\mathbf{I} - \rho[\mathbf{W}_4]_+)^{-1}[\mathbf{M}]_+$	7
XC4	$\mathbf{X}_{\text{stock}}$	$\sigma_Z^2(\mathbf{I} - \rho\mathbf{W}_4)^{-1}$	7
XI4RU	$\mathbf{X}_{\text{stock}}$	(NA, **??**) + $\varepsilon^2\mathbf{I}$	7
XC4RD	$\mathbf{X}_{\text{stock}}$	$\sigma_Z^2(\mathbf{I} - \rho[\mathbf{W}_4 \odot \exp(-\mathbf{D}/\theta_2)]_+)^{-1}[\mathbf{M}]_+$	8
XC4RDS	$\mathbf{X}_{\text{stock}}$	$\sigma_Z^2(\mathbf{I} - \rho[\mathbf{W}_4 \odot \exp(-\mathbf{D}/\theta_2) \odot \exp(-\mathbf{S}/\theta_1)]_+)^{-1}[\mathbf{M}]_+$	9
XC4RDU	$\mathbf{X}_{\text{stock}}$	$\sigma_Z^2(\mathbf{I} - \rho[\mathbf{W}_4 \odot \exp(-\mathbf{D}/\theta_2)]_+)^{-1}[\mathbf{M}]_+ + \sigma_\varepsilon^2\mathbf{I}$	9

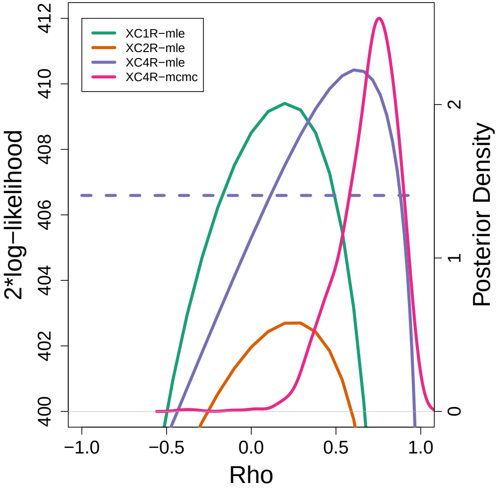
Model Comparison and Selection



Regression

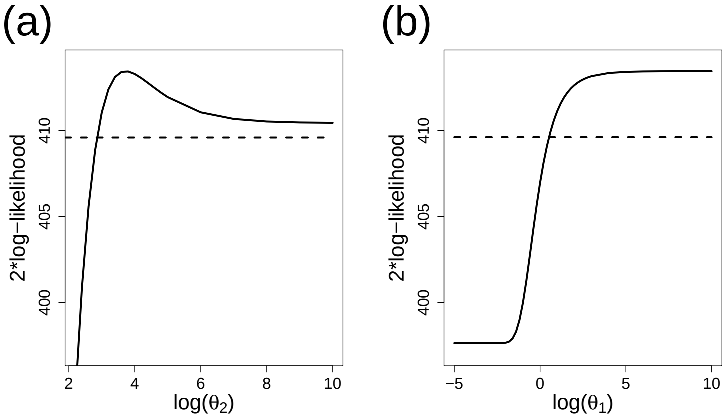
	XU		XC4R-mle		XC4R-mcmc		XC4RD	
Parameter	Est.	Std.Err	Est.	Std.Err	Est.	Std.Err.	Est.	Std.Err
μ	-0.079	0.0225	-0.080	0.0288	-0.082	0.0330	-0.077	0.0290
$\beta_{\text{stock 2}}$	0.048	0.0298	0.063	0.0379	0.063	0.0429	0.058	0.0386
$\beta_{\text{stock 3}}$	0.093	0.0281	0.095	0.0355	0.097	0.0386	0.092	0.0356
$\beta_{\text{stock 4}}$	0.132	0.0279	0.135	0.0346	0.138	0.0406	0.132	0.0346
$\beta_{\text{stock 5}}$	0.084	0.0259	0.093	0.0327	0.096	0.0378	0.089	0.0330

Inference on Autocorrelation Parameter

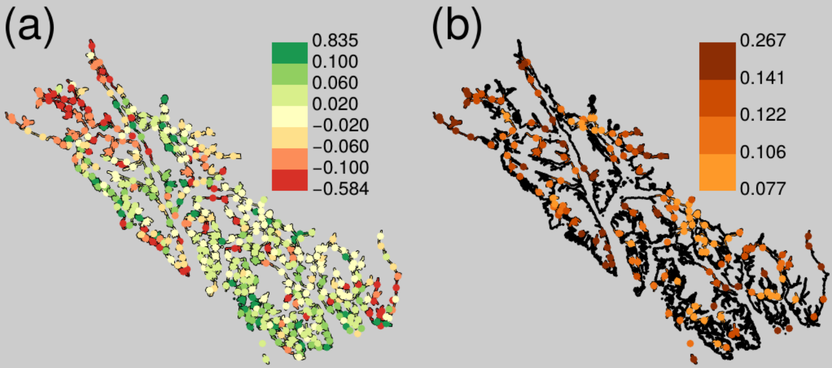


Inference on Connectivity Parameters

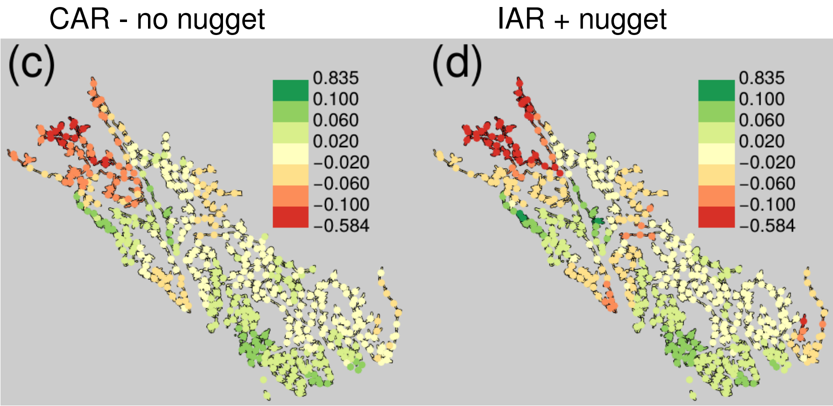
$$\mathbf{W} = \mathbf{N}_i \odot \exp(-\mathbf{S}/\theta_1) \odot \exp(-\mathbf{D}/\theta_2)$$
$$\mathbf{S}_{i,j} = 0 \text{ if } i, j \text{ in same stock, otherwise } 0$$



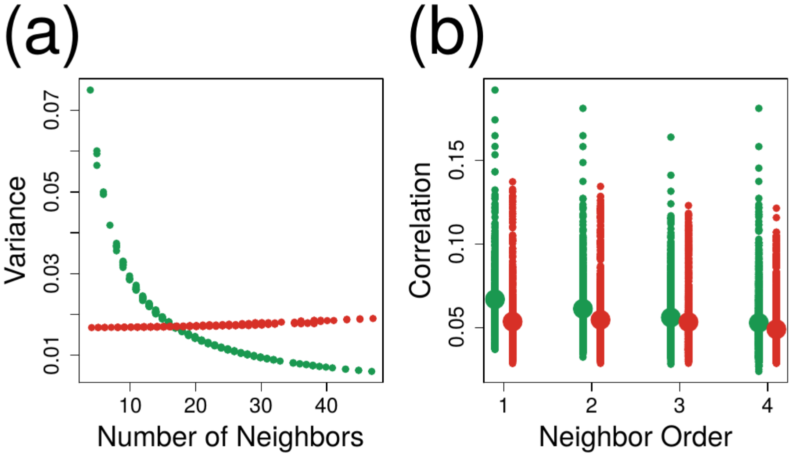
Predictions



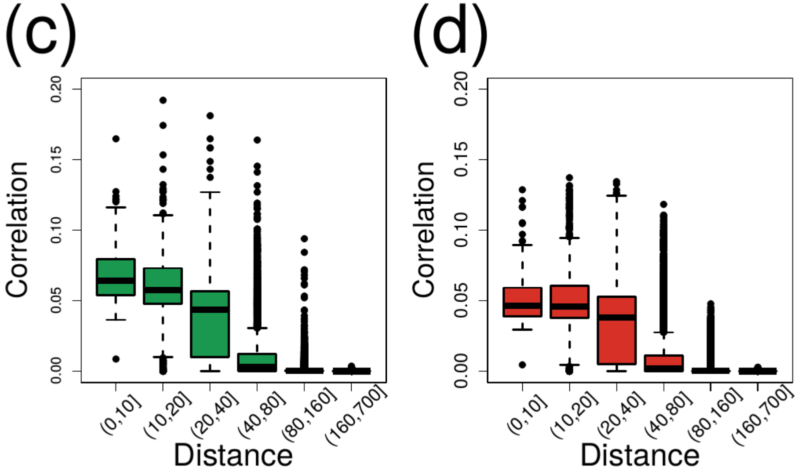
Smoothing


CAR - no nugget
IAR + nugget
(c)
(d)
0.835
0.100
0.060
0.020
−0.020
−0.060
−0.100
−0.584

Variances and Correlations



Fitted Spatial Autocorrelations



Practical Advice

- Models are harder to understand than geostatistical, but they can/should still be used. There many important objectives and applications.
- Important idea is that we are modeling the inverse covariance matrix (precision matrix), and it is essentially equivalent to partial correlations.
- Connectivity ideas can be explored by modeling in the covariance matrix, and inference on parameters obtained through profile likelihoods or posterior distributions. However, they may have large confidence/credibility regions, requiring large sample sizes.
- Row standardization and various neighborhood definitions should be explored.

Practical Advice

- IAR vs. CAR when used hierarchically in generalized linear mixed models. $\mathbf{y} \sim [\mathbf{y}|g(\boldsymbol{\mu}), \delta^2\mathbf{I}]$ and then $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta} + \mathbf{z} + \boldsymbol{\varepsilon}$. To use nugget, ε, or not?
- Model diagnostics, especially for covariance parameters, are especially important for CAR/SAR models.
- Sites without neighbors:

$$\begin{pmatrix} \sigma_I^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma} \end{pmatrix}$$

- Check software carefully!

A Closer Look at Properties

SAR

$$\boldsymbol{\Sigma}_{\text{SAR}} = (\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\Lambda}(\mathbf{I} - \mathbf{B}')^{-1}$$

S1 $(\mathbf{I} - \mathbf{B})$ is nonsingular,

S2 $\boldsymbol{\Lambda}$ is diagonal with positive diagonal elements, and

S3 $b_{i,i} = 0, \ \forall \ i$.

CAR

$$\boldsymbol{\Sigma}_{\text{CAR}} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{M}$$

C1 $(\mathbf{I} - \mathbf{C})$ has positive eigenvalues,

C2 $\mathbf{M}$ is diagonal with positive diagonal elements,

C3 $c_{i,i} = 0, \ \forall \ i$, and

C4 $c_{i,j}/m_{i,i} = c_{j,i}/m_{j,j}, \ \forall \ i, j$.

From SAR to CAR

The easy part, but still most get it wrong!

Let $\mathbf{M} = \mathbf{I}$ and $\mathbf{\Lambda} = \mathbf{I}$

$$(\mathbf{I} - \mathbf{C})^{-1} = [(\mathbf{I} - \mathbf{B})(\mathbf{I} - \mathbf{B}')]^{-1} = (\mathbf{I} - \mathbf{B} - \mathbf{B}' + \mathbf{B}\mathbf{B}')^{-1}$$

so, according to Haining (1990)

$$\mathbf{C} = \mathbf{B} + \mathbf{B}' - \mathbf{B}\mathbf{B}'$$

Repeated many times in textbooks and peer-reviewed papers, but C3 is not satisfied because diagonal of $\mathbf{B}\mathbf{B}'$ not $\mathbf{0}$.

$$\mathbf{C} \neq \mathbf{B} + \mathbf{B}' - \mathbf{B}\mathbf{B}'$$

CAR Theorem

Any positive-definite covariance matrix $\mathbf{\Sigma}$ can be expressed as the covariance matrix of a CAR model $(\mathbf{I} - \mathbf{C})^{-1}\mathbf{M}$, for a unique pair of matrices $\mathbf{C}$ and $\mathbf{M}$.

Outline of Proof

Let $\mathbf{\Sigma}^{-1} = \mathbf{D} - \mathbf{R}$, where $\mathbf{D}$ is diagonal and $\mathbf{R}$ has zeros on the diagonal.

Then $\mathbf{\Sigma}^{-1} = \mathbf{D} - \mathbf{R} = \mathbf{D}(\mathbf{I} - \mathbf{D}^{-1}\mathbf{R}) = \mathbf{M}^{-1}(\mathbf{I} - \mathbf{C})$, where $\mathbf{C} = \mathbf{D}^{-1}\mathbf{R}$ and $\mathbf{M} = \mathbf{D}^{-1}$.

Then $\mathbf{\Sigma}$ can be expressed in CAR form.

Corrected Result from Cressie, 1993, pg. 409

$$\mathbf{M}^{-1}(\mathbf{I}-\mathbf{C}) = \mathbf{I} - 2\mathbf{B} + \mathbf{B}^2$$

$$\begin{aligned} E(Z(u,v)|\{z(k,l):(k,l)\neq(u,v)\}) \\ = (1+2\xi_1^2+2\xi_2^2)^{-1}\{2\xi_1[z(u-1,v)+z(u+1,v)] \\ +2\xi_2[z(u,v-1)+z(u,v+1)] \\ -2\xi_1\xi_2[z(u-1,v-1)+z(u+1,v-1) \\ +z(u+1,v-1)+z(u+1,v+1)] \\ -\xi_1^2[z(u-2,v)+z(u+2,v)] \\ -\xi_2^2[z(u,v-2)+z(u,v+2)]\}. \end{aligned}$$

SAR Theorem

Any positive definite covariance matrix $\mathbf{\Sigma}$ can be expressed as the covariance matrix of a SAR model $(\mathbf{I} - \mathbf{B})^{-1}\mathbf{\Lambda}(\mathbf{I} - \mathbf{B}^T)^{-1}$, for a (non-unique) pair of matrices $\mathbf{B}$ and $\mathbf{\Lambda}$.

Outline of Proof

Let $\mathbf{\Sigma}^{-1} = \mathbf{L}\mathbf{L}'$, where $\mathbf{L}$ is full rank with positive eigenvalues.

Note that $\mathbf{L}$ is *not* unique. For example, a Cholesky or spectral decomposition can be used.

Now let $\mathbf{L} = \mathbf{G} - \mathbf{P}$, where $\mathbf{G}$ is diagonal and $\mathbf{P}$ has zeros on the diagonal.

Then $\mathbf{L}\mathbf{L}' = (\mathbf{G} - \mathbf{P})(\mathbf{G}' - \mathbf{P}') = (\mathbf{I} - \mathbf{P}\mathbf{G}^{-1})\mathbf{G}\mathbf{G}(\mathbf{I} - \mathbf{G}^{-1}\mathbf{P}')$
$= (\mathbf{I} - \mathbf{B}^T)\mathbf{\Lambda}^{-1}(\mathbf{I} - \mathbf{B})$.
where $\mathbf{\Lambda}^{-1} = \mathbf{G}\mathbf{G}$ and $\mathbf{B}^T = \mathbf{P}\mathbf{G}^{-1}$

Corollary

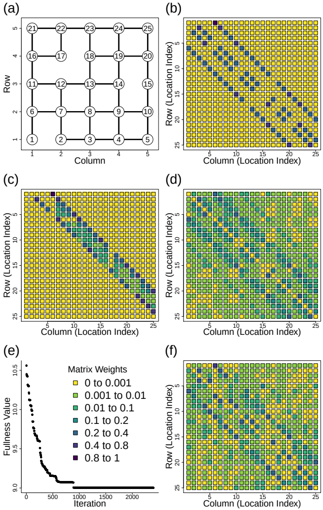
Any $\mathbf{\Sigma}$ can be expressed as one of an infinite number of $\mathbf{B}$ matrices that define the SAR covariance matrix.

$$\mathbf{A}_{h,s}(\theta) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cos(\theta) & 0 & \cdots & 0 & \sin(\theta) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -\sin(\theta) & 0 & \cdots & 0 & \cos(\theta) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

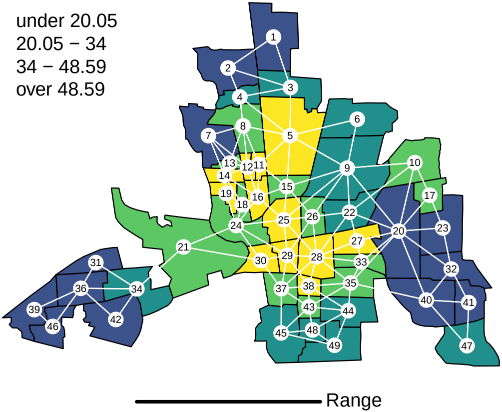
and $\mathbf{\Sigma} = \mathbf{L}\mathbf{L}' = \mathbf{L}\mathbf{A}'_{h,s}\mathbf{A}_{h,s}\mathbf{L}' = \mathbf{L}_*\mathbf{L}'_*$.

CAR to SAR

$$\text{fullness}(\mathbf{B}) = \frac{\sum_{i,j}|b_{i,j}|}{\sqrt{\sum_{i,j} b_{i,j}^2}}$$

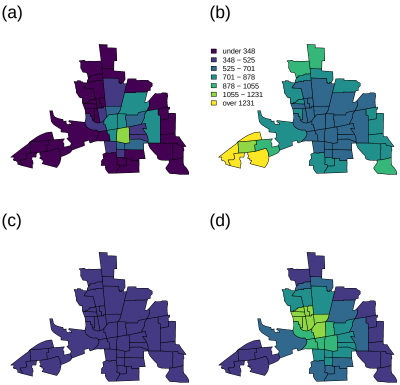


Columbus Crime Data

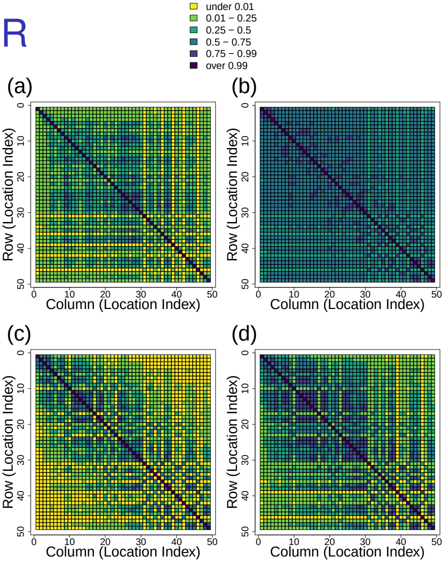


Geostat to CAR

$$\mathbf{\Sigma}_{\mathrm{CAR}} = \sigma^2(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{M} + \delta^2\mathbf{I}$$
$$\mathcal{L}_{\mathrm{un}}(\boldsymbol{\theta};\mathbf{y}) = 388.83$$
$$\mathcal{L}_{\mathrm{rs}}(\boldsymbol{\theta};\mathbf{y}) = 397.25$$
$$\mathbf{\Sigma}_{\mathrm{geo}} = \sigma^2\mathrm{sph}(\mathbf{D}) + \delta^2\mathbf{I}$$
$$\mathcal{L}_{\mathrm{geo}}(\boldsymbol{\theta};\mathbf{y}) = 374.61$$
$$\mathbf{\Sigma}_{\mathrm{C}|\mathrm{g}} = \sigma^2(\mathbf{I} - \rho\mathbf{C}_{\mathrm{geo}})^{-1}\mathbf{M}_{\mathrm{geo}} + \delta^2\mathbf{I}$$
$$\sigma^2 > 0, -1.104 < \rho < 1.013, \delta^2 > 0$$
$$\mathcal{L}_{\mathrm{C}|\mathrm{g}}(\boldsymbol{\theta};\mathbf{y}) = 373.95$$
$$\hat{\sigma}^2_{\mathrm{C}|\mathrm{g}} = 0.941, \hat{\rho}_{\mathrm{C}|\mathrm{g}} = 1.010$$



Geostat to CAR



Alaska Survey Polygons

1308 Polygons with Harbor Seal Counts over 20 Years

STARMA (Cliff, 1975)

$$\mathbf{Z}(t) - \sum_{k=0}^{p} \mathbf{B}_k \mathbf{Z}(t-k) = \varepsilon(t) - \sum_{\ell=0}^{q} \mathbf{E}\varepsilon(t-1)$$

Additive Model

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{V}_{\mathrm{t}|\mathrm{z}_i} + \mathbf{Z}_i + \varepsilon_{\mathrm{i,t}}$$

Spatio-temporal CAR?

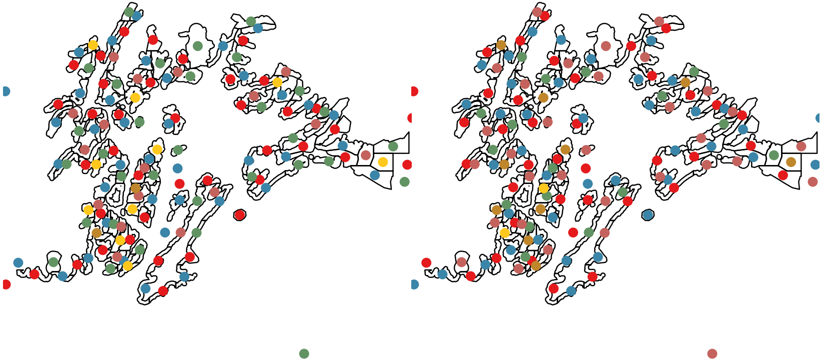
Spatio-temporal CAR

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_{\mathrm{yr1}} \\ \mathbf{z}_{\mathrm{yr2}} \\ \vdots \\ \mathbf{z}_{\mathrm{yr20}} \end{bmatrix}$$

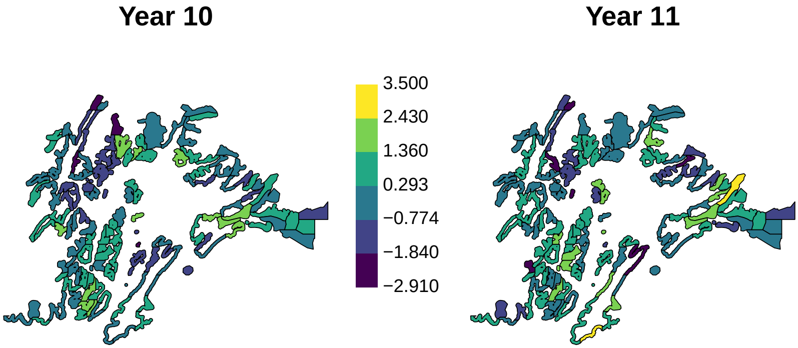
$$\mathbf{C} = \rho \begin{bmatrix} (1-\phi)\mathbf{W}_{+} & \phi\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \frac{\phi}{2}\mathbf{I} & (1-\phi)\mathbf{W}_{+} & \frac{\phi}{2}\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \frac{\phi}{2}\mathbf{I} & (1-\phi)\mathbf{W}_{+} & \frac{\phi}{2}\mathbf{I} & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \phi\mathbf{I} & (1-\phi)\mathbf{W} \end{bmatrix}$$

$0 \le \rho \le 1$ and $0 \le \phi \le 1$

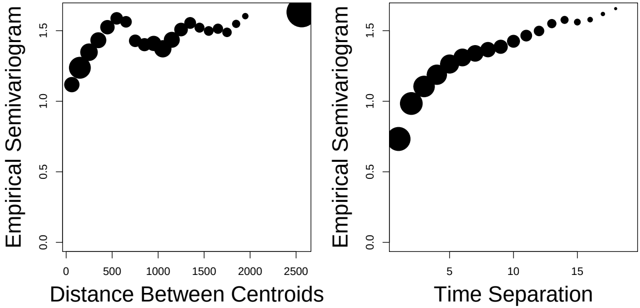
Spatio-temporal Chromatic Coloring



Simulated Data



Simulation Variograms



How to estimate model parameters?
Especially for Poisson/Spatial hierarchical models?

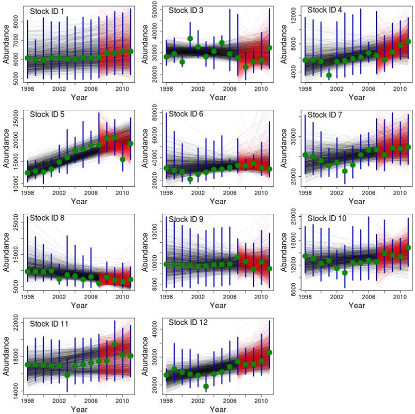
What About Trend?

$\mathbf{f(t)}'\beta \in p(\hat{N}_t | \ldots)$ NOT good

$N_t = \phi N_{t-1} \ldots \in p(\hat{N}_t | \ldots)$ Good

Compute Trend on posterior of

$$\hat{N}_t^{[k]}, \ldots, \hat{N}_{t+\mathcal{I}}^{[k]}$$



Summary

- Review of CAR and SAR Models
  - Model Comparison and Selection (Averaging)
  - Regression (Effect of Explanatory Varibles)
  - Estimation of Autocorrelation Parameters
  - Estimation of Other Network Parameters
  - Prediction of Unsampled Locations
  - Smoothing over Sampled and Unsampled Locations
- Relationships between CAR and SAR
  - Any pos. def. $\boldsymbol{\Sigma}$ can written as CAR or SAR
- Spatio-temporal CAR models?
  - What about trend?