# Monte Carlo Simulation of Selectivity and Maturity at Age in a Length-Based Age-Structured Model

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# Objectives

**Investigate the effects of simulating uncertainty in length at age in an age-structured model**

1. Verify expected outcome of including uncertainty
   (parametric bootstrapping)
2. Discuss remaining questions for simulation methods
   (Rescale $sel_a$ and $mat_a$ to max of 1?)

# Monte Carlo simulation

**Completed Work:**

**Step 1) Model numbers at age with simulated uncertainty in recruitment**
**Step 2) Model sel$_L$ and mat$_L$ and simulate uncertainty in length at age**
**Step 3) Transform simulated uncertainty in length at age into simulated uncertainty in sel$_a$ and mat$_a$**

Ongoing Work:

Step 4) Project numbers at age ahead with fixed (assumed) parameter values for most life history and simulated uncertainty in recruitment and length at age
Step 5) Summarize the distributions of derived variables for stock status
Step 6) Repeat – evaluate model sensitivity to assumed fixed parameter values
Step 7) Risk analysis – summarize results over a range of "what if scenarios"

# Numbers at age

$$N_{a,t+1} = \begin{cases} N_{r,t+1}; & a = r \\ N_{a-1,t} e^{(-M_{a-1} - sel_{a-1} F_t)}; & a > \text{r to } A\text{-}1 \\ (N_{A-1,t} + N_{A,t}) e^{(-M_A - sel_A F_t)}; & a = A \end{cases}$$

(Quinn and Deriso, 1999)

$$C_{a,t} = \frac{sel_a F_t}{M_a + sel_a F_t} N_{a,t} \left(1 - e^{(-M_a - sel_a F_t)}\right)$$

(Quinn and Deriso, 1999;
e.g., Wetzel and Punt, 2011a)

$$\tilde{N}_{a=r,t} = \frac{\alpha E_t}{1 + \beta E_t} e^{\left(-0.5\sigma_R^2 + \varepsilon_t\right)}; \quad \varepsilon_t \sim N\left(0; \sigma_R^2\right)$$

(Quinn and Deriso, 1999;
Brooks and Powers, 2007;
Brooks et al., 2010)

$$E_t = \sum_{a=r}^{A} m_a f_a \rho_{a,t} N_{a,t}$$

(Punt and Walker 1998; cf, Simpfendorfer et al. 2000)

# Length-based sel$_L$ and mat$_L$

$$sel_L = \left(\frac{1}{1-\gamma}\right)\left(\frac{1-\gamma}{\gamma}\right)^{\gamma}\left(\frac{e^{\alpha\gamma(\beta-L)}}{1+e^{\alpha(\beta-L)}}\right)$$

(Thompson, 1994; Sigler, 1999)

$$m_L = \frac{m_\infty}{1+e^{-k(L-\gamma)}}$$

(Quinn and Deriso, 1999)

$$\tilde{L}_a = L_\infty\left(1-e^{-k(a-t_0)}\right)+\varepsilon_{L_a}; \quad \varepsilon_{L_a} \sim N\left(0;\sigma^2_{L_a}\right)$$

(Quinn and Deriso, 1999)

where

$L_\infty$ = asymptotic (maximum) length,
$k$ = the growth rate parameter,
$t_0$ = the age when an individual would have been at L = 0,
$\varepsilon_{L_a} \sim N\left(\mu_{L_a};\hat{\sigma}^2_{L_a}\right)$, where $\mu_{L_a}$ was the predicted LVB length at each age *a* without uncertainty,
$\hat{\sigma}^2_{L_a}$ was the assumed variance in length at each age *a* required to achieve a constant CV.

# Age-length transition matrix

$$\phi_{a,l} = \begin{cases} \Phi\left(\dfrac{L'_{\min} - \tilde{L}_a}{\sigma_{L_a}}\right); & l = 1 \\ \Phi\left(\dfrac{L'_{l+1} - \tilde{L}_a}{\sigma_{L_a}}\right) - \Phi\left(\dfrac{L'_l - \tilde{L}_a}{\sigma_{L_a}}\right); & l > 1 \text{ to } l < A_l \\ 1 - \Phi\left(\dfrac{L'_{\max} - \tilde{L}_a}{\sigma_{L_a}}\right); & l = A_l \end{cases}$$

(Wetzel and Punt, 2011; Methot and Wetzel, in Press)

$$\left.\begin{aligned} sel_a &= \sum_{l=1}^{A_l} \phi_{a,l} sel_l \\ m_a &= \sum_{l=1}^{A_l} \phi_{a,l} m_l \end{aligned}\right\}$$

**? Citations (difficult to interpret)**

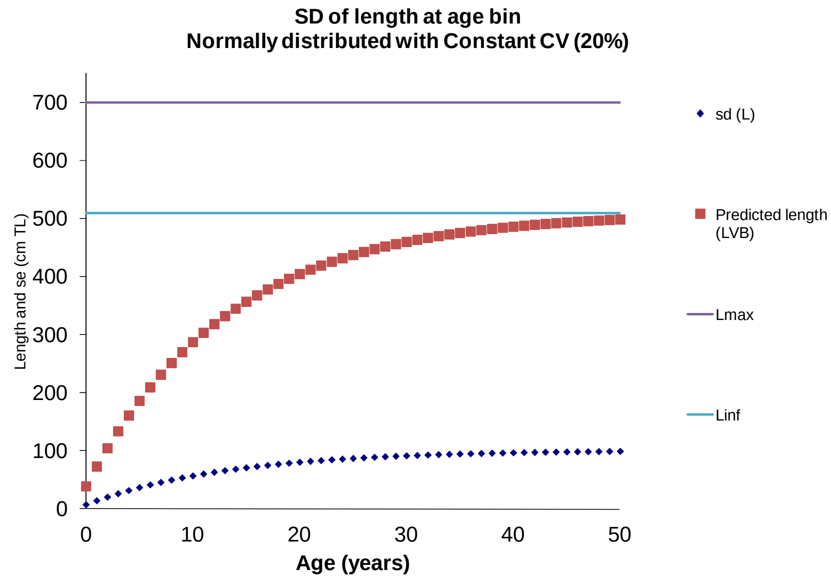
**1.Verify expected outcome (parametric bootstrapping)**

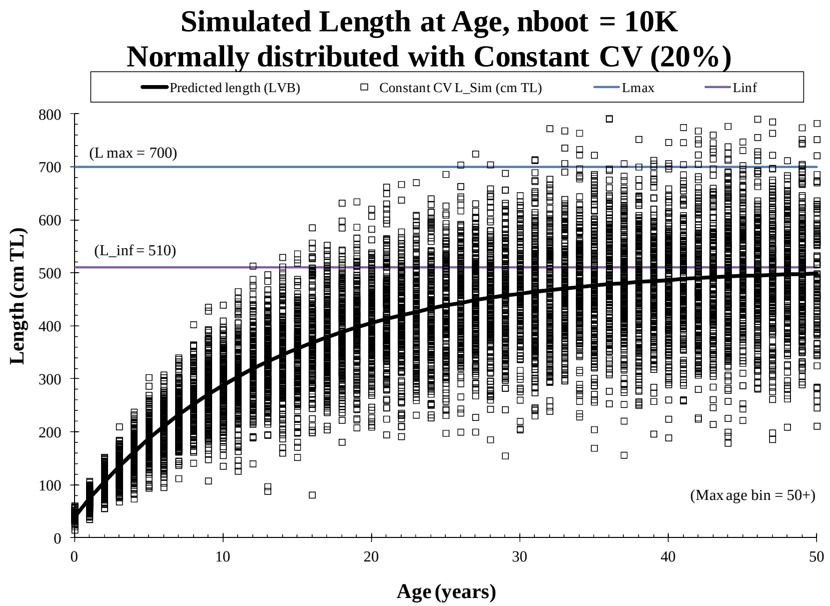
**2.Verify methods (rescale to max of 1?)**

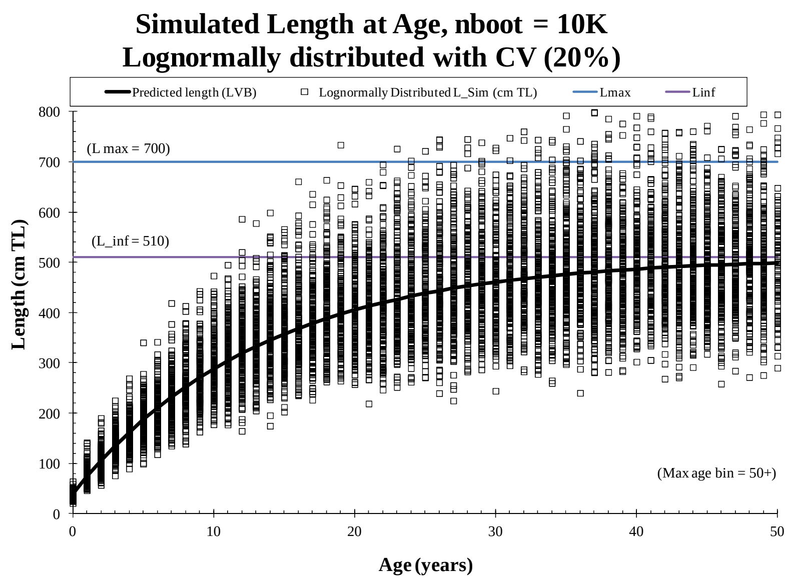
# Verify expected outcome (parametric bootstrap)

- *Draw a random length at each age* (1-50)
  - a) $\tilde{L}_a = L_\infty\left(1-e^{-k(a-t_0)}\right)+\varepsilon_{L_a}; \quad \varepsilon_{L_a} \sim N\left(0;\sigma^2_{L_a}\right), a=(r,r+1,\ldots,A)$
  - b) $\tilde{L}_a = L_\infty\left(1-e^{-k(a-t_0)}\right)e^{\varepsilon_{L_a}}+; \quad \varepsilon_{L_a} \sim N\left(0;\sigma^2_{L_a}\right), a=(r,r+1,\ldots,A)$
- *Determine $sel_L$ and $mat_L$ at simulated length*
  - $sel_{\tilde{L}_a} = \left(\frac{1}{1-\gamma}\right)\left(\frac{1-\gamma}{\gamma}\right)^{\gamma}\left(\frac{e^{\alpha\gamma\left(\beta-\tilde{L}_a\right)}}{1+e^{\alpha\left(\beta-\tilde{L}_a\right)}}\right) \qquad m_{\tilde{L}_a} = \frac{m_\infty}{1+e^{-k\left(\tilde{L}_a-\gamma\right)}}$
  -
- *Assign $sel_L$ and $mat_L$ to each age* (1-50)
  - $sel_{\tilde{a}} = sel_{\tilde{L}_a} \qquad m_{\tilde{a}} = m_{\tilde{L}_a}$
  -
- *Repeat (n=10,000) and summarize*

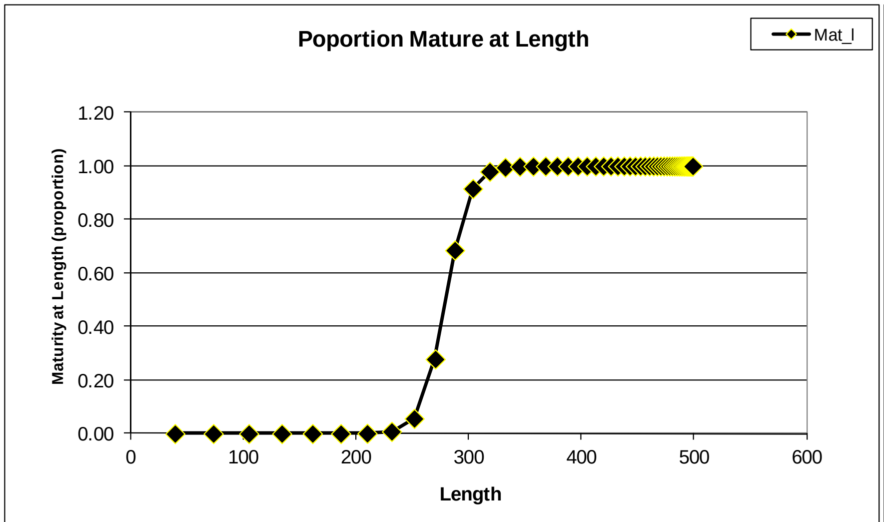
- *Draw a random length at each age* (1-50)

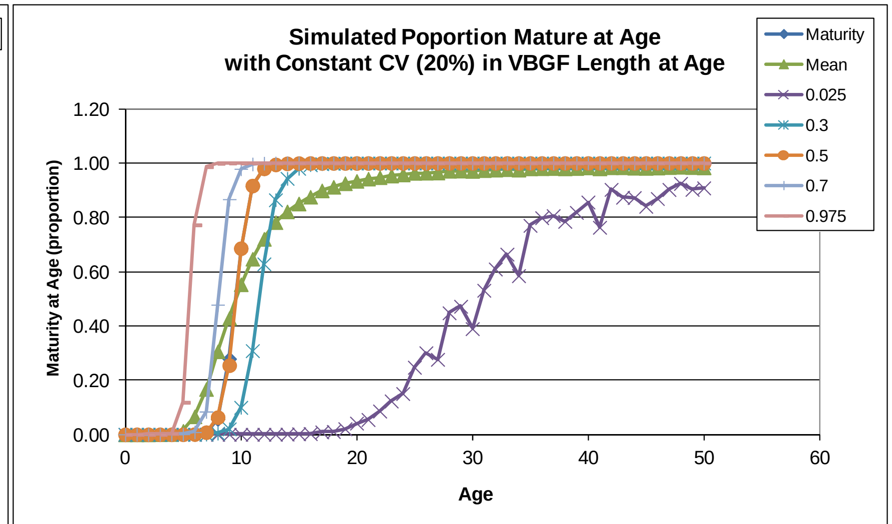


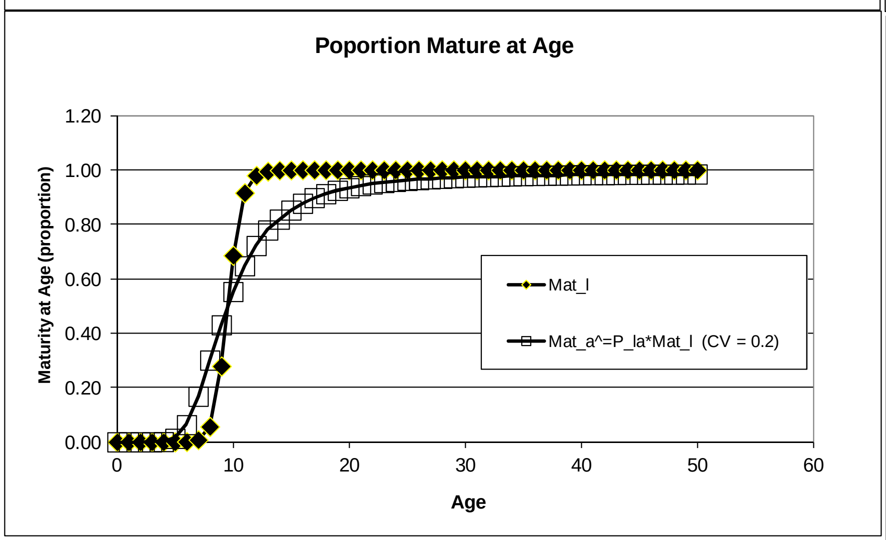



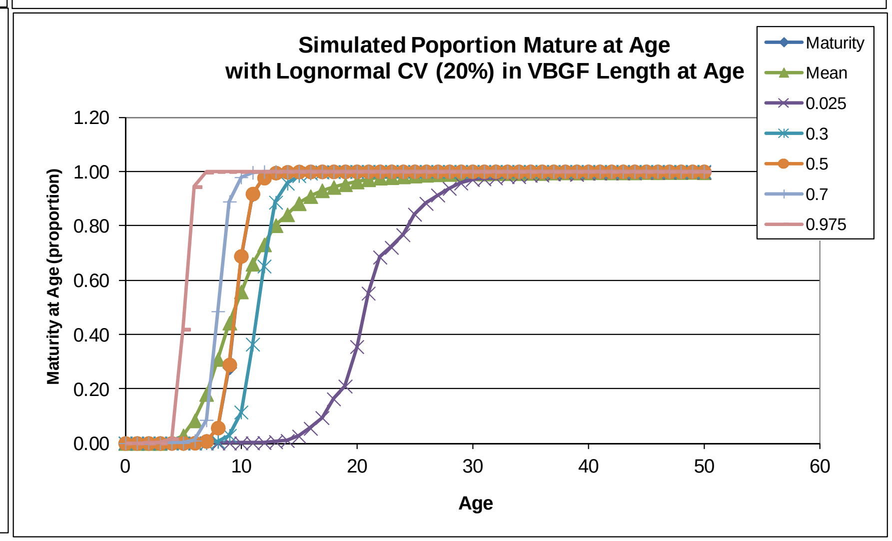



- Matrix $m_a = \sum_{l=1}^{A_l} \phi_{a,l} m_l$
- Bootstrap $m_{\tilde{a}} = m_{\tilde{L}_a}$

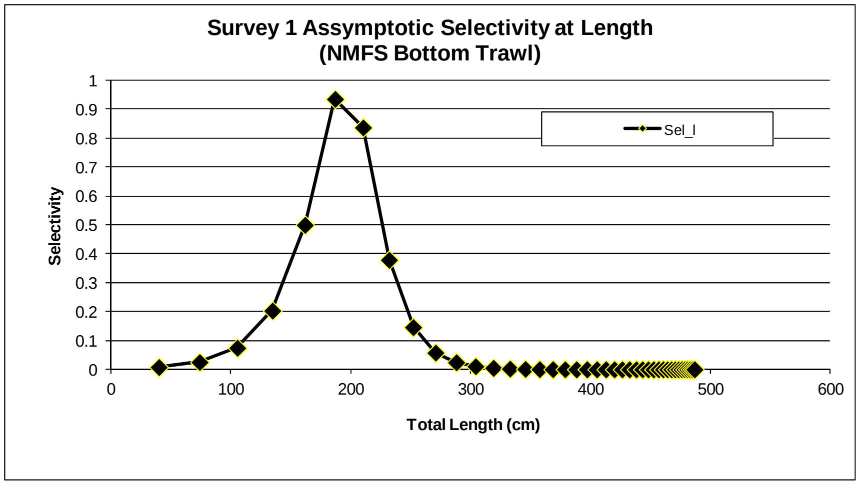


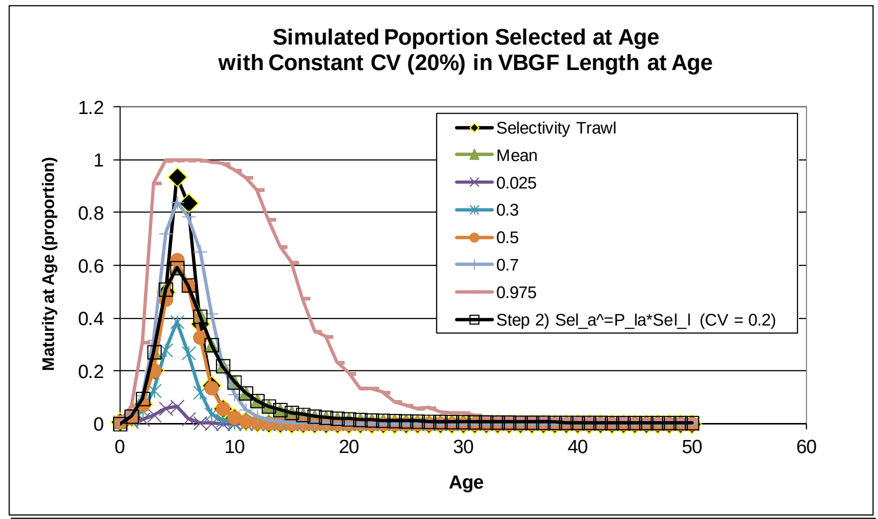



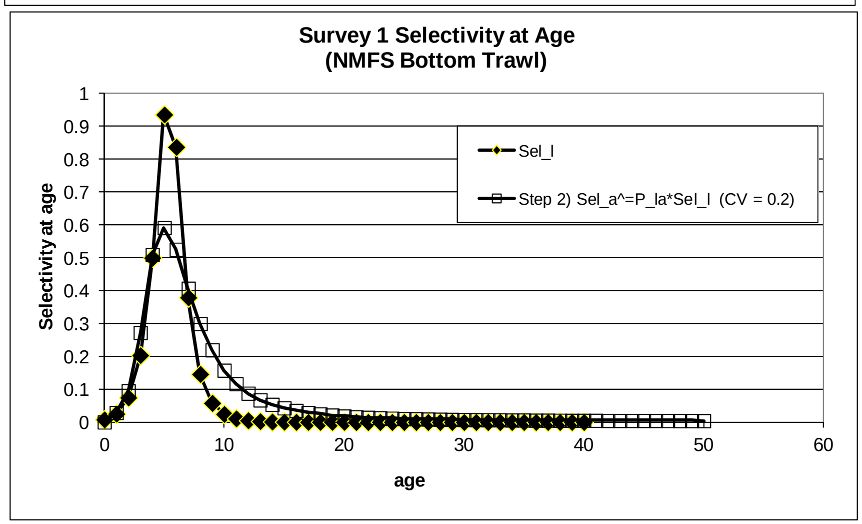



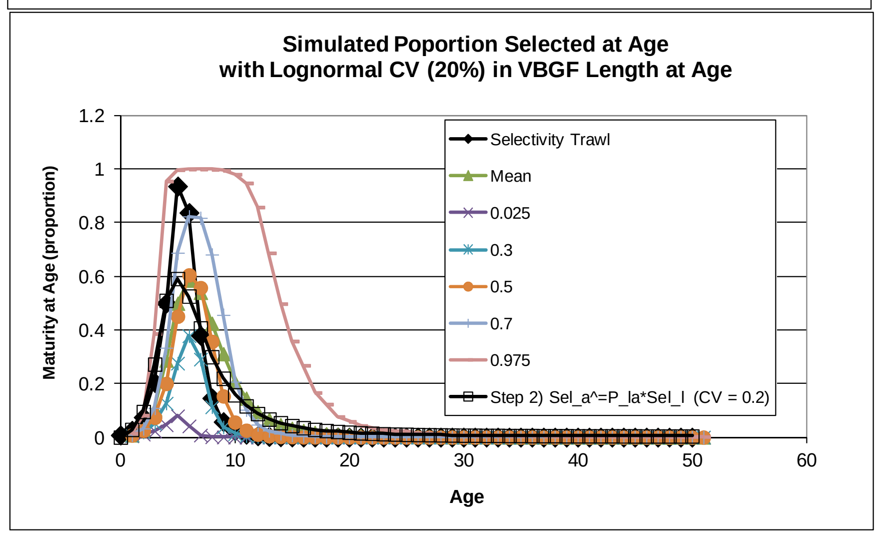



- *Matrix* $sel_a = \sum_{l=1}^{A_l} \phi_{a,l} sel_l$
- *Bootstrap* $sel_{\tilde{a}} = sel_{\tilde{L}_a}$



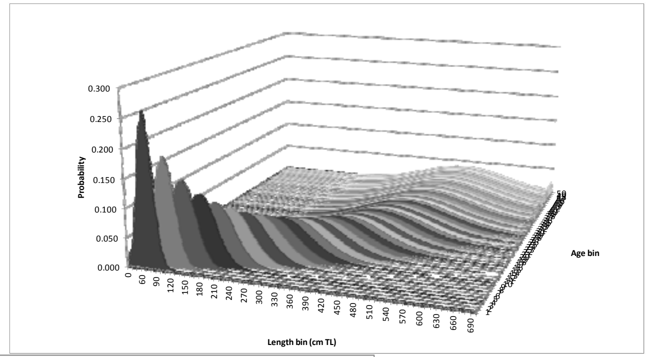



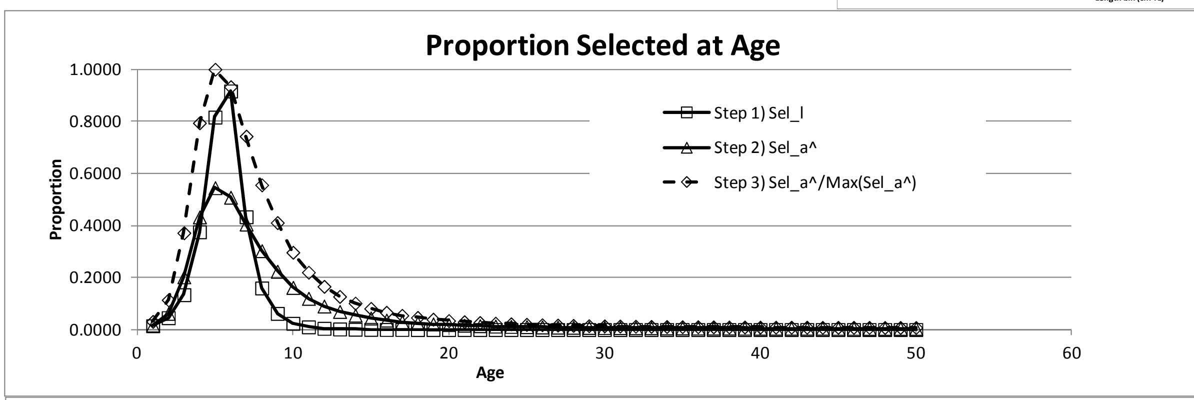
# Conclusions

- **Mean of bootstrap approximated results from transition matrix**
  - $$\text{mean}(sel_{\tilde{L}_a}) \cong sel_a = \sum_{l=1}^{A_l} \phi_{a,l} sel_l$$
- **Median of bootstrap approximated results from back-transformed von Bertalanffy growth curve**
  - $\text{median}(sel_{\tilde{L}_a}) \cong sel_a = sel_{L_a},$
  - where $L_a = L_\infty \left(1 - e^{-k(a-t_0)}\right)$
- To do
  - Rescale to 1? $$sel_a = \sum_{l=1}^{A_l} \phi_{a,l} sel_l$$
    $$sel_a / \max(sel_a)?$$

# Rescale to 1?

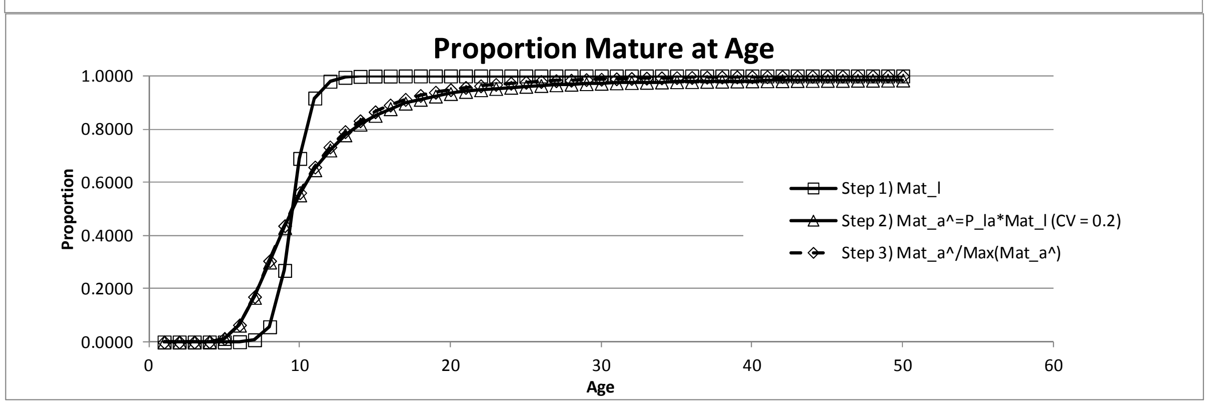
- *Rescale $sel_a$ and $mat_a$ to max of 1?*
  - *Literature not clear*
  - *Rules of thumb?*




$$sel_a = \sum_{l=1}^{A_l} \phi_{a,l} sel_l$$

$$sel_a / \max(sel_a)?$$




$$m_a = \sum_{l=1}^{A_l} \phi_{a,l} m_l$$

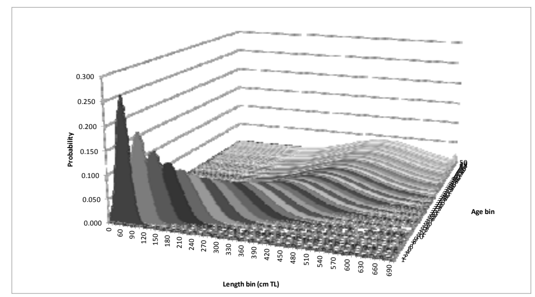
$$m_a / \max(m_a)?$$

# Example: Proportions at length

$$p_{\ell_i,a_j} = \begin{cases} \int_{L_{Max}}^{+\infty} \frac{1}{\sqrt{2\pi}\hat{\sigma}_j} e^{-\left[\frac{(L_i-\bar{L}_j)^2}{2(\hat{\sigma}_j)^2}\right]} dL \text{ For } L_i = L_{Max} \\ \int_{L_i}^{L_i+\Delta_i} \frac{1}{\sqrt{2\pi}\hat{\sigma}_j} e^{-\left[\frac{(L_i-\bar{L}_j)^2}{2(\hat{\sigma}_j)^2}\right]} dL \text{ For } L_{Min} < L_i < L_{Max} \\ \int_{-\infty}^{L_{Min}} \frac{1}{\sqrt{2\pi}\hat{\sigma}_j} e^{-\left[\frac{(L_i-\bar{L}_j)^2}{2(\hat{\sigma}_j)^2}\right]} dL \text{ For } L_i = L_{Min} \end{cases}$$

**e.g., Methot, 1990, and 2000, e.g., their stage 1 model case 4 transition matrix; e.g., Haddon, 2011, their length-to-length transition matrix for a stage based model**

$$\left[\hat{n}_{\ell_1}\hat{n}_{\ell_2}\cdots\hat{n}_{\ell_m}\right] = \left[n_{a_1}n_{a_2}\cdots n_{a_n}\right] \bullet \begin{bmatrix} p_{\ell_1,a_1} & p_{\ell_2,a_1} & \cdots & p_{\ell_m,a_1} \\ p_{\ell_1,a_2} & \ddots & & \\ \vdots & & & \\ p_{\ell_1,a_n} & & & p_{\ell_m,a_n} \end{bmatrix}$$




where each element $\hat{n}_{\ell_i}, i = 1,2,3,\ldots,m$ of $N_\ell$ is calculated as

$$\hat{n}_{\ell_i} = n_{a_1} \times p_{\ell_i,a_1} + n_{a_2} \times p_{\ell_i,a_2} + \ldots + n_{a_n} \times p_{\ell_i,a_n}$$

$$\hat{n}_{\ell_i} = \sum_{j=1}^{n} n_{a_j} \times p_{\ell_i,a_j}$$

Rescale as a proportion by dividing by the sum (~pdf) [standard is to rescale before multiplying by matrix]

# Example: Proportions at age

$$\hat{S}el_a = P_{\ell,a} \bullet Sel_l$$

**(e.g., Methot, 1990, 2000; their stage 2 model – size specific availability; Coleraine Users Manual e.g., their page 8),**

where

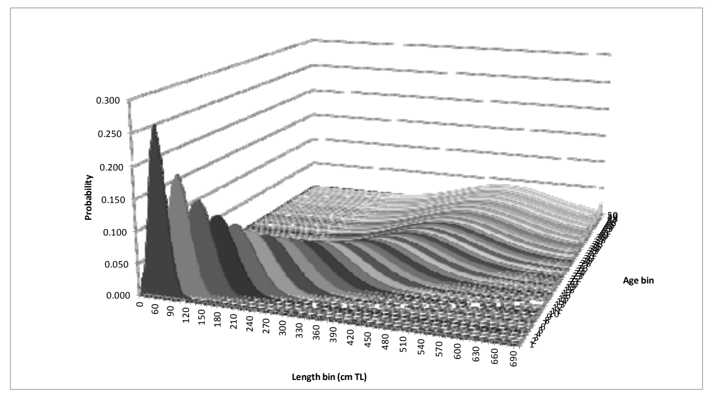
$P_{\ell,a}$ is the $(n \times m)$ length-at-age transition matrix (equation 2),

$Sel_\ell$ is a $(1 \times m)$ vector of the selectivity at length by bin $\ell_i, i = 1,2,3,\ldots,m$, and

$\hat{S}el_a$ is a $(1 \times n)$ vector of the simulated selectivity at age by bin $a_j, j = 1,2,3,\ldots,n$.

An example of the $(n \times 1)$ = $(n \times m)(m \times 1)$ matrix multiplication is

$$\begin{bmatrix} \hat{s}el_{a_1} \\ \hat{s}el_{a_2} \\ \vdots \\ \hat{s}el_{a_n} \end{bmatrix} = \begin{bmatrix} p_{\ell_1,a_1} & p_{\ell_2,a_1} & \cdots & p_{\ell_m,a_1} \\ p_{\ell_1,a_2} & & \ddots & \\ \vdots & & & \\ p_{\ell_1,a_n} & & & p_{\ell_m,a_n} \end{bmatrix} \bullet \begin{bmatrix} sel_{\ell_1} \\ sel_{\ell_2} \\ \vdots \\ sel_{\ell_m} \end{bmatrix}$$




where each element $\hat{s}el_{a_j}, j = 1,2,3,\ldots,n$ of $\hat{S}el_a$ is calculated as

$$\hat{s}el_{a_j} = p_{\ell_1,a_j} \times sel_{l_1} + p_{\ell_2,a_j} \times sel_{l_2} + \ldots + p_{\ell_m,a_j} \times sel_{l_m}$$

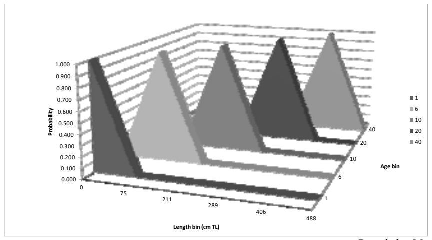
$$\hat{s}el_{a_j} = \sum_{i=1}^{m} p_{\ell_i,a_j} \times sel_{l_i}$$

# Ex. Proportions at age (CV = 0.001)

- *5 age bins and 6 length bins low (CV = 0.0001)*
  - *a = (1, 6, 10, 20, 40) (yrs)*
  - *L= (74+1, 210+1, 288+1, 405+1, 487+1 ) (cm TL)*

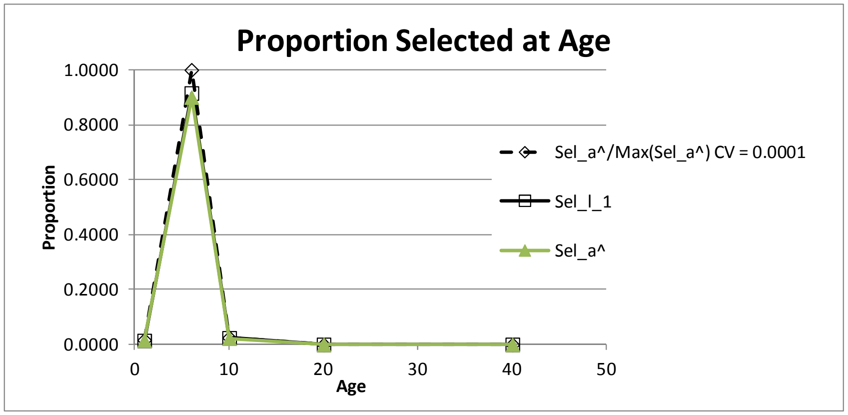
Example 1. Five age bins with low uncertainty: $a_j$ = (1, 6, 10, 20, 40), $\ell_i$ = (74+1, 210+1, 288+1, 405+1, 487+1 cm TL), CV = 0.0001.

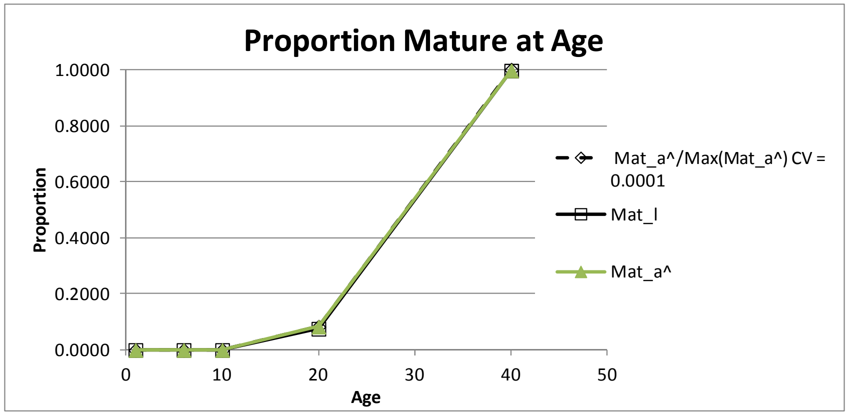
| Assumed true age (a) | Rounded to nearest integer LVB | sd(L) | Check CV | age | Upper bound (e.g., <75) / Lower bound (e.g., >=0) | 75 / 0 | 211 / 75 | 289 / 211 | 406 / 289 | 488 / 406 | 488+ / 488 | sum |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 74 | 0.0074 | 0.0001 | | 1 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 |
| 6 | 210 | 0.0210 | 0.0001 | | 6 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 |
| 10 | 288 | 0.0288 | 0.0001 | | 10 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 1.000 |
| 20 | 405 | 0.0405 | 0.0001 | | 20 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 1.000 |
| 40 | 487 | 0.0487 | 0.0001 | | 40 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 1.000 |




| | a | Pa=PSL | LVB | PSL | sd(L) | Check CV | PL^=P_la*Pa | Rescaled to Max 1 PL^/Max(PL^) |
|---|---|---|---|---|---|---|---|---|
| | 1 | 0.0068 | 73.6003 | 0.0068 | 0.0074 | 0.0001 | 0.0068 | 0.0070 |
| | 6 | 0.9689 | 210.0671 | 0.9689 | 0.0210 | 0.0001 | 0.9689 | 1.0000 |
| | 10 | 0.0238 | 287.8043 | 0.0238 | 0.0288 | 0.0001 | 0.0238 | 0.0246 |
| | 20 | 0.0000 | 405.0422 | 0.0000 | 0.0405 | 0.0001 | 0.0000 | 0.0000 |
| | 40 | 0.0000 | 486.5807 | 0.0000 | 0.0487 | 0.0001 | 0.0000 | 0.0000 |
| Grand Total | Sum | 1.000 | | | | Sum | 1.000 | 1.032 |
| | Max | 0.969 | | | | Max | 0.969 | 1.000 |

| | a | Pa=PML | LVB | PML | sd(L) | Check CV | PL^=P_la*Pa | Rescaled to Max 1 PL^/Max(PL^) |
|---|---|---|---|---|---|---|---|---|
| | 1 | 0.0000 | 73.60027 | 0.0000 | 0.00736 | 0.0001 | 0.0000 | 0.0000 |
| | 6 | 0.0000 | 210.0671 | 0.0000 | 0.021007 | 0.0001 | 0.0000 | 0.0000 |
| | 10 | 0.0000 | 287.8043 | 0.0000 | 0.02878 | 0.0001 | 0.0000 | 0.0000 |
| | 20 | 0.0762 | 405.0422 | 0.0762 | 0.040504 | 0.0001 | 0.0762 | 0.0764 |
| | 40 | 0.9965 | 486.5807 | 0.9965 | 0.048658 | 0.0001 | 0.9965 | 1.0000 |
| Grand Total | Sum | 1.073 | | | | Sum | 1.073 | 1.076 |
| | Max | 0.997 | | | | Max | 0.997 | 1.000 |



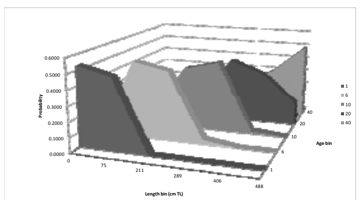



# Ex. Proportions at age (CV = 0.2)

- *5 age bins and 6 length bins low (CV = 0.2)*
  - *a = (1, 6, 10, 20, 40) (yrs)*
  - *L= (74+1, 210+1, 288+1, 405+1, 487+1 ) (cm TL)*

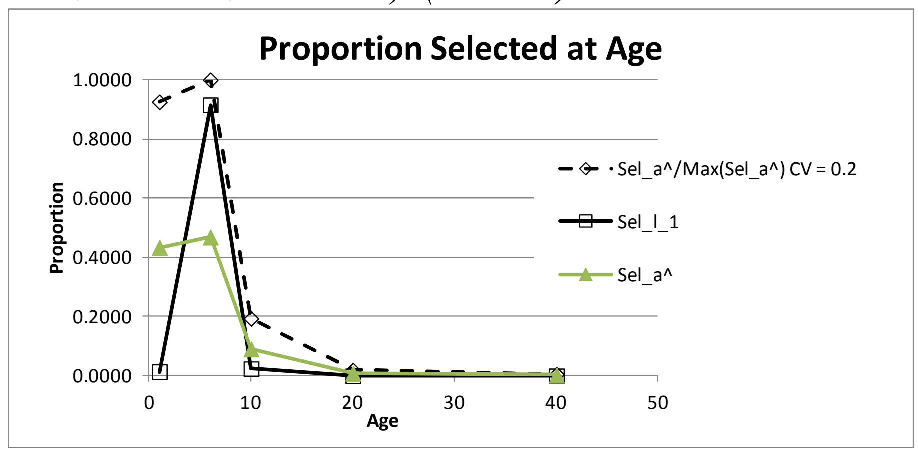
Example 2. Five age bins with moderate uncertainty: $a_j$ = (1, 6, 10, 20, 40), $\ell_i$ = (74+1, 210+1, 288+1, 405+1, 487+1 cm TL), CV = 0.2.

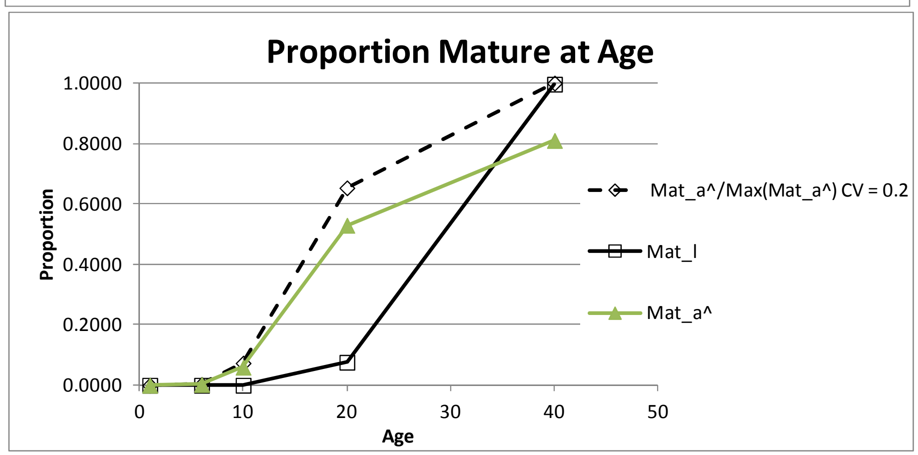
| Assumed true age (a) | Round to nearest integer LVB | sd(L) | Check CV | age | Upper bound (e.g., <75) / Lower bound (e.g., >=0) | 75 / 0 | 211 / 75 | 289 / 211 | 406 / 289 | 488 / 406 | 488+ / 488 | sum |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 74 | 14.7201 | 0.1989 | | 1 | 0.5271 | 0.4729 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 1.000 |
| 6 | 210 | 42.0134 | 0.2001 | | 6 | 0.0007 | 0.5088 | 0.4605 | 0.0300 | 0.0000 | 0.0000 | 1.000 |
| 10 | 288 | 57.5609 | 0.1999 | | 10 | 0.0001 | 0.0904 | 0.4164 | 0.4729 | 0.0199 | 0.0003 | 1.000 |
| 20 | 405 | 81.0084 | 0.2000 | | 20 | 0.0000 | 0.0083 | 0.0678 | 0.4288 | 0.3423 | 0.1528 | 1.000 |
| 40 | 487 | 97.3161 | 0.1998 | | 40 | 0.0000 | 0.0023 | 0.0187 | 0.1817 | 0.3015 | 0.4959 | 1.000 |




| | a | Pa=PSL | LVB | PSL | sd(L) | Check CV | PL^=P_la*Pa | Rescaled to Max 1<br>PL^/Max(PL^) |
|---|---|---|---|---|---|---|---|---|
| | 1 | 0.0068 | 73.6003 | 0.0068 | 14.7201 | 0.2000 | 0.0042 | 0.0084 |
| | 6 | 0.9689 | 210.0671 | 0.9689 | 42.0134 | 0.2000 | 0.4984 | 1.0000 |
| | 10 | 0.0238 | 287.8043 | 0.0238 | 57.5609 | 0.2000 | 0.4561 | 0.9152 |
| | 20 | 0.0000 | 405.0422 | 0.0000 | 81.0084 | 0.2000 | 0.0404 | 0.0810 |
| | 40 | 0.0000 | 486.5807 | 0.0000 | 97.3161 | 0.2000 | 0.0005 | 0.0010 |
| Grand Total | Sum | 1.000 | | | | Sum | 1.000 | 2.006 |
| | Max | 0.969 | | | | Max | 0.498 | 1.000 |

| | a | Pa=PML | LVB | PML | sd(L) | Check CV | PL^=P_la*Pa | Rescaled to Max 1<br>PL^/Max(PL^) |
|---|---|---|---|---|---|---|---|---|
| | 1 | 0.0000 | 73.6003 | 0.0000 | 14.7201 | 0.2000 | 0.0000 | 0.0000 |
| | 6 | 0.0000 | 210.0671 | 0.0000 | 42.0134 | 0.2000 | 0.0029 | 0.0089 |
| | 10 | 0.0000 | 287.8043 | 0.0000 | 57.5609 | 0.2000 | 0.0238 | 0.0728 |
| | 20 | 0.0762 | 405.0422 | 0.0762 | 81.0084 | 0.2000 | 0.2137 | 0.6545 |
| | 40 | 0.9965 | 486.5807 | 0.9965 | 97.3161 | 0.2000 | 0.3265 | 1.0000 |
| Grand Total | Sum | 1.073 | | | | Sum | 0.567 | 1.736 |
| | Max | 0.997 | | | | Max | 0.327 | 1.000 |



# Questions?

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